Appendix G
Using the Sommerfeld–Watson Transformation

G.1 Introduction

The complex spectral number summation technique already has been described briefly in Section 3.7. Here, for illustrative purposes, we outline the technique for computing the scattering from a perfectly reflecting sphere and from a transparent refracting sphere. The technique offers efficient convergence provided that (1) a complete set of poles can be isolated, (2) their residues can be calculated accurately, and (3) the set of poles offers a converging solution. By use of contour integration, we replace the sum for the scattering along the real axis in real-integer spectral number space with another sum in complex spectral number space (see Fig. 3-9). From Eq. (3.12-3), the radial component of the scattered wave in series form is

$$E_r^{(j)} = -\frac{E_o}{x^2}\sum_{l=1}^{\infty} S_l^{(j)}(2l+1)i^{l-1}\xi_l^{+}(x)P_l^1(\cos\theta) \tag{G-1}$$

Here the superscript "(j)" denotes the jth degree scattered wave and $S_l^{(j)}$ is its scattering coefficient. Eq. (3.9-3) gives $S_l^{(0)} = -\left(1+\mathcal{W}_l^- / \mathcal{W}_l\right)/2$, $S_l^{(1)} = n\left(-2i/\mathcal{W}_l\right)^2/2$, $S_l^{(2)} = 2n\mathcal{W}_l^+ / \left(\mathcal{W}_l\right)^3$, etc. For the transverse component, appropriately simplified per discussion after Eq. (3.12-4), a similar expression is obtained:

$$E_\theta^{(j)} = \frac{E_o}{x}\sum_{l=1}^{\infty} S_l^{(j)}\frac{2l+1}{l(l+1)}i^{l-1}\xi_l^{+\prime}(x)P_l^{1\prime}(\cos\theta)\sin\theta \tag{G-2}$$

Let us consider just the radial component in Eq. (G-1).

We define the function

$$F[l]=S_l^{(j)}i^l\,(2l+1)\xi_l^+(x)P_l^1\big(\cos(\theta+\pi)\big) \tag{G-3}$$

Consider the contour integral I^+ in the complex plane along a closed path enclosing the positive real axis and lying at an infinitesimal distance ε above and below the real axis [see Fig. 3-9(a)]:

$$I^+=\frac{1}{2}\left[\int_{0-i\varepsilon}^{\infty-i\varepsilon}\frac{F[l'-1/2]}{\cos(\pi l')}dl'+\int_{\infty+i\varepsilon}^{0+i\varepsilon}\frac{F[l'-1/2]}{\cos(\pi l')}dl'\right] \tag{G-4}$$

We can use the theory of residues in contour integration of analytic functions to evaluate this integral by summing the residues at the simple poles of the integrand, which on the real axis are located only at the half-integer points in l'. Upon summing the residues from the complex contour integrations around each of the poles in Eq. (G-4), one obtains

$$\begin{aligned}I^+&=-i\sum_{\kappa=0}^{\infty}(-1)^{\kappa}F[k]=\sum_{l=1}^{\infty}(-1)^{l-1}iF[l]\\&=\sum_{l=1}^{\infty}S_l^{(j)}i^{l+1}(2l+1)\xi_l^+(x)(-1)^{l-1}P_l^1\big(\cos(\theta+\pi)\big)\\&=-\sum_{l=1}^{\infty}S_l^{(j)}i^{l-1}\,(2l+1)\xi_l^+(x)\,P_l^1\big(\cos\theta\big)=E_r^{(s)}\,\frac{x^2}{E_o}\end{aligned} \tag{G-5}$$

To obtain the result in Eq. (G-5), note that $P_l^1(\cos\ \theta)=(-1)^{l-1}P_l^1(\cos\ (\theta+\pi))$, and also that $F[0]=0$ because $P_0^1(\cos\theta)\equiv 0$. A similar relation holds for $E_\theta^{(s)}$. In this manner, we convert the sum in Eq. (G-1) into the contour integral given in Eq. (G-4).

Next, we need to extend the contour integration in Eq. (G-4) to include the entire real axis so that we can use the vanishing property of the integrand when it is evaluated on a semi-circular arc bounding the upper complex plane whose radius approaches infinity. Therefore, we will need to invoke the symmetry properties of the integrand in this contour integral. From the defining differential equation for the spherical Hankel functions of order $l-1/2$ [see Eq. (3.6-2)], it follows that these functions must be either symmetric or anti-symmetric in l. It is easily shown from the defining equation for the Bessel function that

$$\xi_{l-1/2}^{\pm}(x)=(-1)^l\,\xi_{-l-1/2}^{\pm}(x) \tag{G-6}$$

It follows from Eq. (3.5-11) that all of the Wronskian scattering terms, $\mathcal{W}_{l-1/2}^{\pm}$ and $\mathcal{W}_{l-1/2}$, each of which involves products of a pair of spherical Hankel functions, are symmetric in l. Therefore, it follows from Eq. (3.5-11) that all scattering coefficients of spectral number $l-1/2$, $S_{l-1/2}^{(0)}$, $S_{l-1/2}^{(1)}$, $S_{l-1/2}^{(2)}, \cdots$, are symmetric in l. Next, from the defining differential equation for the Legendre functions, we have

$$P_{l-1/2}^{1}(\cos\theta) = P_{-l-1/2}^{1}(\cos\theta) \tag{G-7}$$

Also, $P_0^1(\cos\theta) = P_{-1}^1(\cos\theta) \equiv 0$. Noting that $2(l-1/2)+1 = 2l$, it follows that

$$(-2l)i^{-l-1/2} = (-1)^{l+1}(2l)i^{l-1/2} \tag{G-8}$$

Assembling all of the parts, we see that

$$\frac{F[-l-1/2]}{\cos(-\pi l)} = -\frac{F[l-1/2]}{\cos(\pi l)}, \quad F[0] = F[-1] = 0 \tag{G-9}$$

Now, the contour integral I^- enclosing the negative real axis is to be taken in the same counter-clockwise sense that also applied to I^+ in Eq. (G-4). Therefore, we obtain for I^-

$$I^- = \frac{1}{2}\left[\int_{0+i\varepsilon}^{-\infty+i\varepsilon} \frac{F[l'-1/2]}{\cos(\pi l')} dl' + \int_{-\infty-i\varepsilon}^{0-i\varepsilon} \frac{F[l'-1/2]}{\cos(\pi l')} dl'\right] \tag{G-10}$$

Upon making a change of integration variable to $v = -l'$ and applying the anti-symmetry property in Eq. (G-9), the resulting contour integrals in Eq. (G-10) become identical with those in Eq. (G-4). Hence, $I^- = I^+$. It follows that we may extend our contour integration in Eq. (G-4) to enclose the entire real axis. We obtain

$$E_r^{(s)} = \frac{E_o}{4x^2}\left[\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{F[l-1/2]}{\cos(\pi l)} dl + \int_{\infty+i\varepsilon}^{-\infty+i\varepsilon} \frac{F[l-1/2]}{\cos(\pi l)} dl\right] \tag{G-11}$$

The same symmetry holds for $E_\theta^{(s)}$.

It is convenient to change from the spectral number variable l in Eq. (G-11) to the argument of the Airy functions y because we will be using the asymptotic forms for the Hankel functions in terms of Airy functions and we will be concerned with the zeros of certain combinations of these functions. The defining relationships between y and l are given by Eqs. (3.8-2) and (3.8-3) (here $l = \nu - 1/2$), which can be greatly simplified in our case where

$x_o >> 1$ to the near-linear relationships given in Eq. (3.13-15). In this case, Eq. (G-11) becomes

$$\left.\begin{array}{l} E_r^{(s)} = \dfrac{E_o K_{x_o}}{4x^2}\left[\displaystyle\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{F[l(y)-1/2]}{\cos(\pi l(y))}dy + \int_{\infty+i\varepsilon}^{-\infty+i\varepsilon} \frac{F[l(y)-1/2]}{\cos(\pi l(y))}dy + \mathrm{O}\left[x_o^{-1}\right]\right] \\ \quad l = x_o + yK_{x_o} \end{array}\right\} \tag{G-12}$$

and similarly for $E_\theta^{(s)}$.

Next, we deform this closed contour encompassing the entire real axis for y into one that excludes the real axis as interior points but encompasses the upper complex plane [see Fig. 3-9(b)]. For Hankel functions of the first kind, it can be shown that the integral along the outer boundary of this deformed contour, whose radius approaches infinity, is zero. Thus, the integral around the closed path of the deformed contour may be evaluated in terms of the residues at the poles of the integrand located anywhere in the complex plane except on the real axis. The potential advantage of this approach may be a much more rapidly converging series in the complex plane than the original series along the positive real axis.

To obtain expressions for $E_r^{(s)}$ and $E_\theta^{(s)}$, we must find the poles of $F[l(y)-1/2]/\cos(\pi l(y))$ other than those on the real axis and sum up the residues at these poles. Let y_κ, $\kappa = 1,2,\cdots$ define the location of those poles of $F[l(y)-1/2]/\cos(\pi l(y))$ in the complex plane away from the real axis. From Eqs. (3.5-11) and (G-3), we see that these poles occur at the zeros of $\mathcal{W}_{l-1/2}$. Moreover, Eq. (3.5-11) also shows that when $F[l-1/2]$ represents the scattering coefficients for the jth-degree scattered wave, it has poles of order $j+1$ at these zeros. Thus, $F[l-1/2]$ for an externally reflected wave has only simple poles; $F[l-1/2]$ for the primary wave, which is refracted twice as it passes through the sphere without any internal reflections, has second-order poles; and so on.

Let us write

$$\frac{F[l(y)-1/2]}{\cos(\pi l(y))} = \frac{U[y]}{V[y]} \tag{G-13}$$

where $U[y]$ has no poles for $\mathrm{Im}[y] > 0$ and is not zero at the zeros of $V[y]$, which occur at the points $y = y_\kappa$, $\kappa = 1,2,\cdots$. If $F[l-1/2]$ has poles of order m at these points, then $V[y_\kappa] = V'[y_\kappa] = \cdots = V^{(m-1)}[y_\kappa] = 0$, but $V^{(m)}[y_\kappa] \neq 0$. From the theory of analytic functions, it follows that the residue for $U[y]/V[y]$ at a pole of order m is given by

$$\oint_{y=y_\kappa} \frac{U[y]}{V[y]} dy = 2\pi i \frac{mU^{(m-1)}}{V^{(m)}}\Bigg|_{y=y_\kappa} \qquad \text{(G-14)}$$

G.2 Application to a Perfectly Reflecting Sphere

Let us first consider the case of the externally reflected wave from a large, perfectly reflecting sphere, which has only simple poles. From residue theory, it follows from Eqs. (G-12) and (G-14) that the radial component of the scattered field is given by

$$E_r^{(s)} = \frac{i\pi E_o}{2x^2} K_{x_o} \sum_{\kappa=1}^{\infty} \frac{U[y_\kappa]}{V'[y_\kappa]} \qquad \text{(G-15)}$$

Here the sequence $y_\kappa, \kappa = 1,2,\cdots y_\kappa, \kappa = 1,2,\cdots$ defines the locations in the complex plane where $V[y]$ is zero.

Recalling the asymptotic forms for the scattering coefficients, we have from Eqs. (3.9-3) and (3.17-1)

$$-S_{l-1/2}^{(0)} = \frac{1}{2}\left(1 + \frac{\mathcal{W}_{l-1/2}^{-}}{\mathcal{W}_{l-1/2}}\right) \underset{n\to\infty}{\rightarrow} \frac{\psi'_{l-1/2}(x_o)}{\xi'^{+}_{l-1/2}(x_o)} \rightarrow \frac{\mathrm{Ai}'[y]}{\mathrm{Ai}'[y] - i\,\mathrm{Bi}'[y]} \qquad \text{(G-16)}$$

where y and l are given by Eq. (3.13-15) with ν replaced by $l+1/2$, i.e., $y = K_{x_o}^{-1}(l+1/2-x_o)+\cdots$ and $l+1/2 = x_o + K_{x_o} y + \cdots$. It follows that U and V are given by

$$\left.\begin{aligned} U(y) &= -2i^{\mu}\, l\xi_\mu^{+}(x)\,\mathrm{P}_\mu^1(\cos(\theta+\pi))\,\mathrm{Ai}'[y]/\cos(\pi l) \\ V(y) &= \mathrm{Ai}'[y] - i\,\mathrm{Bi}'[y] \\ \mu &= l - 1/2 \end{aligned}\right\} \qquad \text{(G-17)}$$

For $x_o >> 1$, the asymptotic forms for the Hankel functions in terms of Airy functions [see Eq. (3.8-1)] apply. Also, for an observer some distance from the reflecting sphere, $x >> x_o >> 1$, the asymptotic forms for the Airy functions in terms of complex exponential functions apply [see Eq. (3.8-10)]. In this case,

$$i^{\mu}\xi_\mu^{+}(x) \underset{x>x_o>>1}{\rightarrow} \left(\frac{x}{D_l}\right)^{1/2} \exp\left(i\left(D_l + l\theta_l - \frac{\pi}{2}\right)\right),\ \mu = l - \frac{1}{2} \qquad \text{(G-18)}$$

where D_l and θ_l, now complex, are still defined in Eq. (3.10-3) (see Fig. 3-14). Similarly, the product $\mathrm{P}_\mu^1[\cos(\theta+\pi)]/\cos(l\pi)$ has the asymptotic form

$$\frac{\mathrm{P}_\mu^1\left[\cos(\theta+\pi)\right]}{\cos(l\pi)} \underset{\mu>>1}{\rightarrow} \left(\frac{2\mu}{\pi\sin\theta}\right)^{1/2} \frac{\cos\left[l(\theta+\pi)+\pi/4\right]}{\cos(l\pi)},\ \mu = l - 1/2 \tag{G-19}$$

The zeros of $\left[\mathrm{Ai}'[\hat{y}] - i\,\mathrm{Bi}'[\hat{y}]\right]$ only lie in the first quadrant of the complex plane where $\mathrm{Re}[y] > 0$ and $\mathrm{Im}[y] > 0$. It follows that $\mathrm{Re}[l] = x_o + \mathrm{Re}[y]K_{x_o} >> 1$ and $\mathrm{Im}[l] = \mathrm{Im}[y]K_{x_o} >> 1$. Therefore, $\exp(i\pi l) \rightarrow 0$, and we can simplify Eq. (G-19) to

$$\frac{\mathrm{P}_\mu^1\left[\cos(\theta+\pi)\right]}{\cos(l\pi)} \rightarrow \left(\frac{2\mu}{\pi\sin\theta}\right)^{1/2} \exp\left(-i\left(l\theta + \frac{\pi}{4}\right)\right),\ \mu = l - \frac{1}{2} \tag{G-20}$$

It follows that

$$U(y) \underset{\substack{|y|>>\sim 2/K \\ x>>x_o>>1}}{\rightarrow} 4\left(\frac{x^3 \sin^3\theta_l}{2\pi i \cos\theta_l \sin\theta}\right)^{1/2} \exp\left[i\left(D_l + l(\theta_l - \theta)\right)\right]\mathrm{Ai}'[y] \tag{G-21}$$

When this asymptotic form for $U(y)$ plus $V(y)$ is used in Eq. (G-11), it is easily shown that the result is virtually identical to the scattering integral for $E_r^{(S_o)}$ using the stationary phase approach as given in Eq. (3.12-5) (with the phasor $\exp(i\Phi^+)$ deleted). Thus, we could have deformed the scattering integral in Eq. (3.12-5) directly into one that spanned the upper complex plane to arrive at Eq. (G-15) without enduring the foregoing discussion leading up to Eq. (G-15).

To evaluate Eq. (G-15), we need the zero crossings of $\left(\mathrm{Ai}'[\hat{y}] - i\,\mathrm{Bi}'[\hat{y}]\right)$, which are exhibited in Fig. G-1. They lie only along the straight line defined by $y = \beta\exp(i\pi/3)$ in the positive half of the complex plane, and their values, given by β_κ, $\kappa = 1, 2\cdots$, are shown in this figure. Because of the high sensitivity of l and μ for large x_o to changes in y, $|\partial l/\partial y| = |\partial\mu/\partial y| \doteq K_{x_o} \approx 475$, high precision is required in determining the values of y_κ in order to obtain accurate values for the phasor part of $U[y_\kappa]$. Asymptotic formulas for obtaining the roots of $\mathrm{Ai}'[y] - i\,\mathrm{Bi}'[y] = 0$ are found in [1]. The values $-\beta_\kappa$, $\kappa = 1, 2\cdots$ are also the zeros of $\mathrm{Ai}'[y]$ along the negative real axis [see Eq. (G-25)]. From the defining differential equation for the Airy function, $z'' = xz$, it follows that to calculate the residue at the pole we have $V'(y) = y\left(\mathrm{Ai}[y] - i\,\mathrm{Bi}[y]\right)$. We have, therefore, all of the parts required to calculate the electric field of the reflected wave at the low Earth orbiting (LEO) satellite as a function of θ using the theory of residues and Eq. (G-15).

In order for the summed series in Eq. (G-15) to converge practicably, $U(y_\kappa)$ must approach zero with increasing κ in an efficient way. Inspection of Eq. (G-21) shows that this convergence question hinges on the behavior of the imaginary part of the complex phase term $\Phi_l = D_l + l(\theta_l - \theta)$. By expanding Φ_l in powers of $(l - x_o)$, we obtain

$$\Phi_l = \Phi_{x_o} + (\theta_o - \theta)K_{x_o} y + \frac{(K_{x_o} y)^2}{2D_o} \tag{G-22}$$

A sign change in $d(\mathrm{Im}[\Phi_l])/d(\mathrm{Im}[y])$ occurs at $\mathrm{Im}[y] = D_{x_o}(\theta - \theta_o)/K_{x_o} \approx 200(\theta - \theta_o)$, where $(\theta - \theta_o)$ is expressed in milliradians. Therefore, when the angular position of the LEO is above the geometric shadow boundary, i.e., $(\theta - \theta_o) > 0$, ever so slightly, one obtains a very slowly converging series, which is impracticable. On the other hand, for $(\theta - \theta_o) \le 0$, the series converges rapidly. The latter is, of course, the shadow region where no stationary phase points exist. Thus, two methods for summing the spectral series, the stationary phase technique and the contour integration technique in the complex spectral number plane, complement each other to some extent.

G.3 Application to a Refracting Sphere

The extension to a refracting sphere with a finite index of refraction is straightforward. Here one replaces the derivatives of the Airy functions shown in Eqs. (G-17) and (G-21) with the corresponding Wronskian forms from Eq. (3.5-11). For example, for the external reflected component, $j = 0$, and one obtains

$$\left.\begin{aligned} U(y) &= \left(\frac{x^3 \sin^3\theta_l}{2\pi i \cos\theta_l \sin\theta}\right)^{1/2} \exp\left[i\left(D_l + l(\theta_l - \theta)\right)\right]\left(\mathcal{W}_l^- + \mathcal{W}_l\right) \\ V(y) &= \mathcal{W}_l,\ \ l + \frac{1}{2} = x_o + K_{x_o} y + \cdots \end{aligned}\right\} \tag{G-23a}$$

For the refracted wave passing through the sphere without internal reflections, $j = 1$, and $U(y)$ and $V(y)$ become

$$\left.\begin{aligned}
U(y) &= 4\left(\frac{x^3 \sin^3\theta_l}{2\pi i \cos\theta_l \sin\theta}\right)^{1/2} \exp\Big[i\big(D_l + l(\theta_l - \theta)\big)\Big] \\
V(y) &= \left(\mathcal{W}_l\right)^2, \ \ l + \frac{1}{2} = x_o + K_{x_o} y + \cdots
\end{aligned}\right\} \tag{G-23b}$$

which has poles of order two at the zeros of $\mathcal{W}_l$.

In the case of either $j = 0$ or $j = 1$, we need to isolate the zeros of $\mathcal{W}_l$, as defined in Eq. (3.5-11). Using the Airy function asymptotic forms for the Hankel functions, it follows from Eqs. (3.8-1)–(3.8-4), that the Wronskian scattering forms in Eq. (3.5-11) are given by

$$\left.\begin{aligned}
\mathcal{W}_l^{\pm} &= \pi n^{1/2}\Big(n(\mathrm{Ai}'[y] \mp i\,\mathrm{Bi}'[y])(\mathrm{Ai}[\hat{y}] \mp i\,\mathrm{Bi}[\hat{y}]) \\
&\qquad - n^{-1}(\mathrm{Ai}[y] \mp i\,\mathrm{Bi}[y])(\mathrm{Ai}'[\hat{y}] \mp i\,\mathrm{Bi}'[\hat{y}])\Big) + \mathrm{O}\Big[N, \nu^{-5/3}\Big] \\
\mathcal{W}_l &= \pi n^{1/2}\Big(n(\mathrm{Ai}'[y] - i\,\mathrm{Bi}'[y])(\mathrm{Ai}[\hat{y}] + i\,\mathrm{Bi}[\hat{y}]) \\
&\qquad - n^{-1}(\mathrm{Ai}[y] - i\,\mathrm{Bi}[y])(\mathrm{Ai}'[\hat{y}] + i\,\mathrm{Bi}'[\hat{y}])\Big) + \mathrm{O}\Big[N, \nu^{-5/3}\Big] \\
y &= \nu^{2/3}\zeta\left[\frac{\nu}{x_o}\right] \doteq K_{x_o}^{-1}\left(\nu - x_o\right) \\
\hat{y} &= \nu^{2/3}\zeta\left[\frac{\nu}{n x_o}\right] \doteq n^{-1/3} y - N x_o K_{n x_o}^{-1}\left(1 + \mathrm{O}\Big[\left(n x_o\right)^{-1}\Big]\right)
\end{aligned}\right\} \tag{G-24}$$

The zero points of $\mathcal{W}_l$ also are shown in Figure G-1 for refractivity values ranging continuously from $N x_o = 83\pi$ down to $N x_o = \pi/10$. As N increases, we note from Eq. (G-24) that $\left(\mathcal{W}_l + \mathcal{W}_l^-\right)/\mathcal{W}_l \to 2\,\mathrm{Ai}'[y]/\left(\mathrm{Ai}'[y] - i\,\mathrm{Bi}'[y]\right)$; therefore, the zero points of $\mathcal{W}_l$ should approach the zeros on the line $\mathrm{Im}[y] = \tan[\pi/3]\mathrm{Re}[y]$, the zeros for the perfectly reflecting case.

However, as N grows large, the form for $\mathcal{W}_l$ given in Eq. (G-24) loses numerical precision. Although the magnitude of y may be small, $|\hat{y}| \doteq \left|n^{-1/3} y - N x_o K_{n x_o}^{-1}\right|$ can be large if $N x_o >> K_{x_o}$. Because y is complex, $|\mathrm{Ai}[\hat{y}]|$ and $|\mathrm{Bi}[\hat{y}]|$ can grow very large. Equation (G-24), which involves differences between these terms, can lose numerical precision. This can be alleviated to a certain extent by noting several identities involving the Airy functions in the complex plane [1]. These are given by

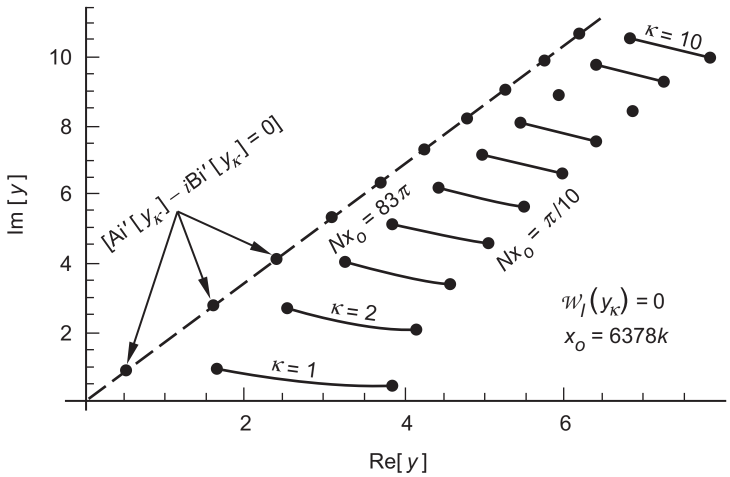

Fig. G-1. Location of zeros for Ai′[*y*] – *i*Bi′[*y*] **and also for** $\mathcal{W}_l$**.**

$$\left.\begin{aligned}\mathrm{Ai}[y]\pm i\,\mathrm{Bi}[y]&=2(p^{\pm})^{1/2}\,\mathrm{Ai}\left[yp^{\mp}\right]\\ \mathrm{Ai}'[y]\pm i\,\mathrm{Bi}'[y]&=2(p^{\mp})^{1/2}\,\mathrm{Ai}'\left[yp^{\mp}\right]\\ \mathrm{Ai}[y]+p^{+}\,\mathrm{Ai}\left[yp^{+}\right]+p^{-}\,\mathrm{Ai}[yp^{-}]&=0\\ \mathrm{Bi}[y]+p^{+}\,\mathrm{Bi}\left[yp^{+}\right]+p^{-}\,\mathrm{Bi}[yp^{-}]&=0\\ p^{\pm}&=\exp\left[\pm 2\pi i/3\right]\end{aligned}\right\}\qquad\text{(G-25)}$$

and

$$\left.\begin{aligned}&\mathscr{W}\left[\mathrm{Ai}\left[yp^{+}\right],\mathrm{Ai}[yp^{-}]\right]=i(2\pi)^{-1},\ \ \mathscr{W}\left[\mathrm{Ai}[y],\mathrm{Ai}\left[yp^{\pm}\right]\right]=\pm\left(p^{\mp}\right)^{1/4}(2\pi)^{-1}\\ &\mathscr{W}\left[\mathrm{Ai}[y],\mathrm{Bi}[y]\right]=\pi^{-1}\end{aligned}\right\}\qquad\text{(G-26)}$$

where $\mathscr{W}[X,Y]$ is the Wronskian of X and Y. Thus, Eq. (G-24) can be rewritten in the form

$$\left.\begin{aligned}\mathcal{W}_l^{\pm}&=4\pi n^{1/2}\left(n\,\mathrm{Ai}[yp^{\pm}]\,\mathrm{Ai}'[\hat{y}p^{\pm}]-n^{-1}\,\mathrm{Ai}'[yp^{\pm}]\,\mathrm{Ai}[\hat{y}p^{\pm}]\right)\\ \mathcal{W}_\ell&=4\pi n^{1/2}\left(np^{-}\,\mathrm{Ai}[yp^{+}]\,\mathrm{Ai}'[\hat{y}p^{-}]-n^{-1}p^{+}\,\mathrm{Ai}'[yp^{+}]\,\mathrm{Ai}[\hat{y}p^{-}]\right)\end{aligned}\right\}\qquad\text{(G-27)}$$

When N is small, even though $Nx_o >> K_{x_o}$, we can factor out the n^{-1} term in Eq. (G-27) and expand the remaining n^2 term in powers of N. The problem term is the first involving the zeroth power of N. We can write this term in the form

$$p^- \operatorname{Ai}[yp^+]\operatorname{Ai}'[\hat{y}p^-] - p^+ \operatorname{Ai}'[yp^+]\operatorname{Ai}[\hat{y}p^-] = \operatorname{Ai}[yp^+]^2 \frac{d}{dy}\left(\frac{\operatorname{Ai}[\hat{y}p^-]}{\operatorname{Ai}[yp^+]}\right) \quad \text{(G-28)}$$

Also, in this case when $\operatorname{Im}[\hat{y}p^-] \neq 0$, one can use the exponential asymptotic forms for $\operatorname{Ai}[\hat{y}p^-]$ and $\operatorname{Bi}[\hat{y}p^-]$ given in Eqs. (3.8-7) and (3.8-8).

For decreasing N, the zeros of $\mathcal{W}_l$ for a fixed value of κ drift logarithmically downward and to the right in Fig. G-1. For very small N, $Nx_o >> K_{x_o}$, the locations of the zeros are given by the asymptotic condition

$$\pi N K_{x_o}^2 \left(y\left(\operatorname{Ai}[y]^2 + \operatorname{Bi}[y]^2\right) - \operatorname{Ai}'[y]^2 - \operatorname{Bi}'[y]^2 \right) = i \quad \text{(G-29)}$$

Figure G-1 shows that, for a fixed value of N, the zeros of $\mathcal{W}_l$ migrate upward from the real axis for increasing integer values of the index κ. As was the case for the perfectly reflecting sphere, here it also can be readily shown that $U(y) \to 0$ exponentially for increasing $\operatorname{Im}[y_\kappa]$ when $(\theta - \theta_o) \le 0$. In this case the individual contribution of the zeros to the scattering coefficients diminishes exponentially as their distance above the real axis increases.

G.4 Aggregate Scattering

For the aggregate scattering coefficient, which is given by Eq. (3.5-15b), we see that its poles are located where $\mathcal{W}_l + \mathcal{W}_l^+ = 0$. From Eqs. (G-24) and (G-26), it follows that

$$\mathcal{W}_l + \mathcal{W}_l^+ \doteq \begin{cases} 2\pi n^{1/2}\left(n \operatorname{Ai}[\hat{y}](\operatorname{Ai}'[y] - i\operatorname{Bi}'[y]) - n^{-1}\operatorname{Ai}'[\hat{y}](\operatorname{Ai}[y] - i\operatorname{Bi}[y])\right) \\ \text{or} \\ 4\pi n^{1/2}\left((p^+)^{1/2} n \operatorname{Ai}[\hat{y}]\operatorname{Ai}'[yp^+] - (p^-)^{1/2} n^{-1}\operatorname{Ai}'[\hat{y}]\operatorname{Ai}[yp^+]\right) = \\ 4\pi (np^+)^{-1/2}\operatorname{Ai}[\hat{y}]\left(\operatorname{Ai}[\hat{y}]\dfrac{d}{dy}\left(\dfrac{\operatorname{Ai}[yp^+]}{\operatorname{Ai}[\hat{y}]}\right) + \left(2N + N^2\right)p^+ \operatorname{Ai}'[yp^+]\right) \end{cases} \quad \text{(G-30)}$$

Unfortunately, the zeros of $\mathcal{W}_l + \mathcal{W}_l^+$ are located in an infinite string along and slightly above the negative real y-axis. Figure G-2 shows the first several zero points for two fixed values of N. Here, contrary to the case shown in Fig. G-1, $\operatorname{Im}[y_\kappa]$ either decreases or does not increase sharply with increasing κ. Also, the numerator term $U(y)$ for S_l does not converge to zero for large values of κ. Therefore, numerical convergence problems result in attempting to evaluate the aggregate scattering field using the poles of S_l in complex spectral number

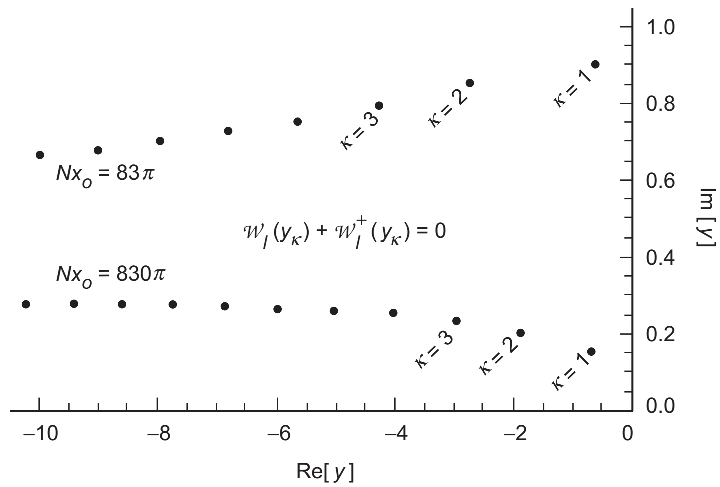

Fig. G-2. Zeros of $\mathcal{W}_l + \mathcal{W}_l^+$ in the complex plane, from Eq. (G-30)

space. One can sum up the contributions from the individual jth-degree scattered waves up to some specified truncation degree using the zeros of $\mathcal{W}_l$ and the residues from the jth-order pole approach outlined earlier. This would require successive numerical differentiation of the appropriate expressions for $U(y)$ and $V(y)$ per the requirements given in Eq. (G-14) for a jth-order pole.

A numerical integration approach aided by the stationary phase technique, which has been used extensively in this monograph, seems simpler for smaller magnitudes of N, roughly $N <\sim 2K_{x_o}^{-2}$. But it requires high precision and also dense sampling to avoid aliasing, and when multiple rays are involved, it also requires care in the vector addition of the field contributions from these separate stationary phase neighborhoods in spectral number. For larger magnitudes of N, the complex spectral number summation technique is preferable in the region where $(\theta - \theta_o) \leq 0$, even with the aforementioned numerical precision problems and the convergence problem for the aggregate scattering coefficient. Also, near shadow boundaries, numerical integration of the scattering integrals when N is large becomes a struggle because of the high accelerations in phase in the scattering coefficients and the large number of stationary points in spectral number.

The bottom line is that there seems to be no free lunch in wave theory for calculating accurately in all regimes the complete field vector of an electromagnetic wave that has passed through a transparent refracting sphere. Parabolic equation methods such as the multiple phase screen approach to propagate the wave through a refracting sphere is perhaps the most promising, but multiple internal reflections from the sphere ($j = 2,3\cdots$) would require careful treatment. For limb sounding from a LEO, a thin-screen/scalar diffraction model offers an attractive alternative, provided in-screen caustics are avoided.

Reference

[1] M. Abramowitz and I. Stegun, eds., *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables,* National Bureau of Standards Series 55, Washington, DC, 1964.